Optimal investment and consumption while allowing for terminal debts

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Agenda

- Merton's optimal investment problem in continuous-time setting (Merton (1969, 1971))
- SAHARA utility (Chen&Pelsser&Vellekoop (JET, 2011))
- Optimal consumption and investment allowing for terminal debts (Chen&Vellekoop (2014))
- Conclusion

Introduction

- Optimal investment problem in continuous-time setting dates back to Merton (1969, 1971)
- An agent chooses a dynamic investment strategy that maximizes his expected utility.
  - Dynamic programming approach and static martingale approach
- The solution to optimal investment problems strongly depends on the risk preferences of the optimizing agent
  - ▷ Realistic but tractable choices for the description of these preferences are therefore important $\Rightarrow$ constant ARA (exponential utility) or constant RRA (power utility)
- To obtain closed-form solutions for optimal investment problems, the assumption of Geometric Brownian asset dynamics is often coupled to assumptions of CARA and CRRA

Financial market

- Assume that we are in a market with a traded risky asset S and a riskfree asset B satisfying

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= r B_t dt \end{aligned}$$

- An agent who can only trade in these two assets in a self-financing way starting with initial wealth x_0 will have a wealth process

$$\begin{aligned} dX_t =& \theta_t \frac{dS_t}{S_t} + (X_t - \theta_t) \frac{dB_t}{B_t} \\ =& (rX_t + \theta_t(\mu - r))dt + \theta_t \, \sigma \, dW_t, \quad X_0 = x_0. \end{aligned}$$

  - ▷ θ_t: amount invested in the risky asset
  - ▷ After the initial time, any fluctuations in the underlying assets can be neutralized by rebalancing strategy in such a way no further gains or losses result

Utility maximization

- The agent tries to maximize the expected utility of his wealth at a time $T > 0$ so the problem can be stated as

$$\max_{\theta \in \mathcal{X}(x_0)} \mathbf{E}[U(X_T)]$$

where $\mathcal{X}(x_0)$ is the class of all possible self-financing strategies in this market when starting from an initial capital x_0.

- Merton relies on dynamic programming approach
$\Rightarrow$ HJB PDE of indirect utility function $u(t,x) = \mathbf{E}[U(X_T^*)|X_t = x]$

$$\begin{cases} \max_{\theta} \left[u_t + (rx + (\mu - r)\theta)u_x + \frac{1}{2}(\sigma\theta)^2 u_{xx} \right] = 0 \\ u(T,x) = U(x) \end{cases}$$

Optimal investment

- The optimal amount invested in the risky asset (Merton portfolio) is given by

$$\theta^* = \frac{(\mu - r)/\sigma^2}{-u_{xx}/u_x}$$

- Substituting the optimal θ^* back to HJB results in the PDE

$$\begin{cases} u_t + rx\, u_x - \frac{1}{2}\lambda^2 \frac{u_x^2}{u_{xx}} = 0 \\ u(T, x) = U(x) \end{cases}$$

where $\lambda := (\mu - r)/\sigma$ is the Sharpe ratio

▷ The closed-form solution (optimal terminal wealth, strategy and indirect utility) can only be obtained in very few cases: CARA (exponential) and CRRA (power) utility

Dual formulation (martingale approach)

- The optimal investment problem:

$$\max_{X_T} \mathbf{E}[U(X_T)] \quad \text{s.t.} \quad \mathbf{E}[H_T X_T] = x_0.$$

  - ▷ In a complete market setting, there is a unique state price density process (the pricing kernel)

$$H_t = \exp\left\{-rt - \frac{1}{2}\lambda^2 t - \lambda W_t\right\},$$

  - ▷ there exists also a unique equivalent martingale (also known risk-neutral) probability measure Q defined by the Radon-Nikodym derivative $\frac{dQ}{dP}|_{\mathcal{F}_t} = e^{rt} H_t$.
  - ▷ the time zero price of a stochastic payoff X_T at some time point T is given by

$$\mathbf{E}^Q[e^{-rT} X_T] = \mathbf{E}[H_T X_T]$$

Way of solutions

- Step 1: The solution to optimal terminal wealth has been derived in the literature (see e.g. Cox and Huang (1989) and Karatzas et al. (1987)) and is given by

$$X_T^* = I(\Lambda H_T)$$

  where $I = (U')^{-1}$ and Λ is determined in terms of x_0 by solving the budget constraint $\mathbf{E}[H_T I(\Lambda H_T)] = x_0$.

- Step 2: Determine $X_t^* = \mathbf{E}[H_T/H_t X_T^* | \mathcal{F}_t]$ as a function of S_t and use $\partial X_t^* / \partial S_t$ to determine the strategy, the indirect utility

Chen&Pelsser&Vellekoop: Motivations to SAHARA utility

- Sometimes there can be marked differences between absolute or relative risk aversion levels for low and high wealth
  - Agents might be more inclined to be less risk averse once their wealth levels are sufficiently low
  - In a crisis, an economic agent might go for more risky portfolios with the hope that high risks will pay off finally, although it occurs with a low probability.
- Sometimes it is useful to work with utility functions defined on the whole real line, but closed-form solutions for the associated optimal investment problems have only been formulated for CARA utilities.

SAHARA Utility

- Symmetric asymptotic hyperbolic absolute risk aversion (SAHARA) utility functions:
  - ▷ Their domain is the whole real line
  - ▷ They allow ARA to be non-monotone
  - ▷ There exists a level of wealth, which we call the default point, where the absolute risk aversion reaches a finite maximal value
    - ◊ The intuition behind this construction is the belief that agents try to avoid default, and hence risk aversion increases when we approach the point from above, but that beyond the default point we see decreasing risk-aversion for increasingly negative levels of wealth.

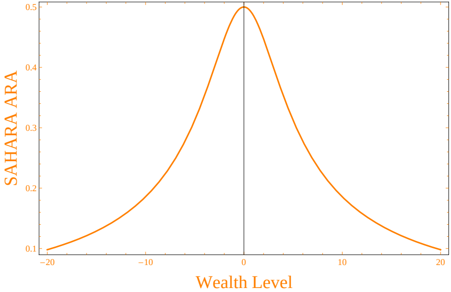
Definition of a SAHARA utility

- Definition: A utility function U with domain $\mathbb{R}$ is of the SAHARA class if its absolute risk aversion function $A(x) = -U''(x)/U'(x)$ is well-defined on its entire domain and satisfies

$$A(x) = \frac{\alpha}{\sqrt{\beta^2 + (x-d)^2}}$$

  - ▷ $\beta > 0$ (scale parameter)
  - ▷ $\alpha > 0$ (risk-aversion parameter)
  - ▷ $d \in \mathbb{R}$ (threshold wealth). We take $d = 0$ from now on

Symmetric Asymptotic Hyperbolic Absolute Risk Aversion



$$A(x) = \frac{\alpha}{\sqrt{\beta^2 + (x-d)^2}}$$

with $\alpha > 0, \beta > 0, d = 0$

- Defined for the real line
- Very tractable
- Preferences with CARA and CRRA as limiting cases
  - $\beta \to 0$ for $x > 0$, SAHARA utility $\to$ power utility.
  - $\alpha = \beta\gamma$ and $\beta \to \infty$, SAHARA utility $\to$ exponential utility

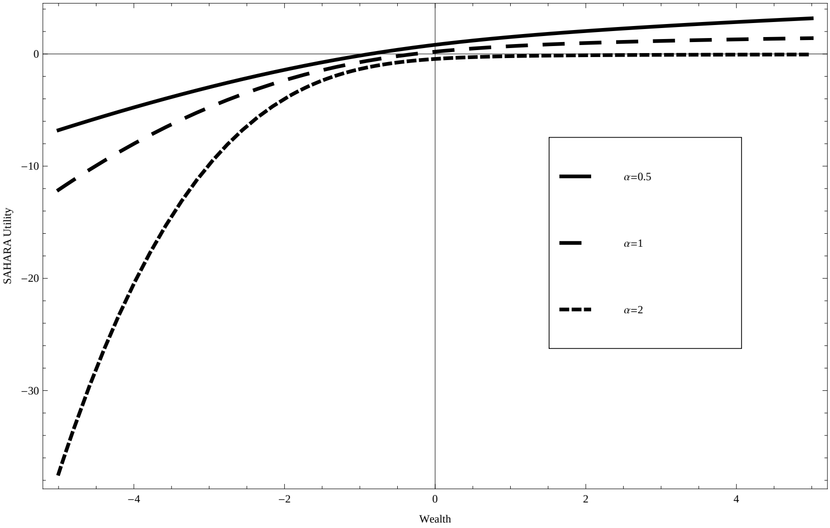
Properties of a SAHARA utility

- Let U be a SAHARA utility function with scale parameter $\beta > 0$ and risk aversion parameter $\alpha > 0$. Then
  - There exists constants c_1 and c_2 such that $U(x) = c_1 + c_2 \hat{U}(x)$ with

$$\hat{U}(x) = \begin{cases} -\frac{1}{\alpha^2-1}\left(x + \sqrt{\beta^2 + x^2}\right)^{-\alpha}\left(x + \alpha\sqrt{\beta^2 + x^2}\right) & \alpha \neq 1 \\ \frac{1}{2}\ln(x + \sqrt{\beta^2 + x^2}) + \frac{1}{2}\beta^{-2}x(\sqrt{\beta^2 + x^2} - x) & \alpha = 1 \end{cases}$$

where the domain is R in both cases.

SAHARA Utility with $\beta = 1$ and $\alpha = 0.5$, 1, 2



..Properties of a SAHARA utility

- The convex dual $\tilde{U}(y) = \sup_{x \in \mathbb{R}} (U(x) - xy)$ and inverse marginal utility function $I = (U')^{-1}$ are

$$\tilde{U}(y) = \begin{cases} \frac{1}{2}\left(\frac{\beta^2 y^{1+1/\alpha}}{1+1/\alpha} - \frac{y^{1-1/\alpha}}{1-1/\alpha}\right) & \alpha \neq 1 \\ \frac{1}{4}(\beta^2 y^2 - 1) - \frac{1}{2}\ln y & \alpha = 1 \end{cases}$$

and

$$I(y) = \frac{1}{2}(y^{-1/\alpha} - \beta^2 y^{1/\alpha}) \ = \ \beta \sinh\left(-\frac{1}{\alpha}\ln y - \ln \beta\right)$$

with domain $y \in \mathbb{R}^+$.

Solutions for SAHARA

- Under SAHARA, the optimal terminal wealth is given by

$$X_T^* = \frac{1}{2}\left(\frac{S_T}{C}\right)^{\lambda/(\alpha\sigma)} - \frac{1}{2}\beta^2\left(\frac{S_T}{C}\right)^{-\lambda/(\alpha\sigma)}$$

where $C > 0$ is a constant which depends on the initial wealth x_0 and other known parameters.

- The optimal investment strategy in terms of wealth that should be invested in the risky asset at time $t \in [0, T]$

$$\theta_t^* = \frac{\lambda}{\alpha\sigma}\sqrt{(X_t^*)^2 + b(t)^2}, \qquad b(t) = \beta e^{-(r-\frac{1}{2}\lambda^2/\alpha^2)(T-t)},$$

Solutions for SAHARA

- The indirect utility function associated with this strategy $u(t,x) = \mathbf{E}[U(X_T^*)|X_t = x]$ is given by the following expression, which solves the HJB PDE mentioned before:

$$u(t,x) = \frac{1}{1-\alpha^2} e^{-(r+\frac{1}{2}\lambda^2/\alpha)(\alpha-1)(T-t)} \left(x + \sqrt{x^2 + b^2(t)}\right)^{-\alpha} \left(x + \alpha\sqrt{x^2 + b^2(t)}\right).$$

Calibration

- The calibrated parameters should implement a desired credit-rating, by means of a prescribed maximal probability of default
- Let ε be the desired maximal probability at time T that the optimal wealth will fall below 0:

$$P\left(X_T^* < 0\right) \leq \varepsilon.$$

- The restriction on the default probability is equivalent to

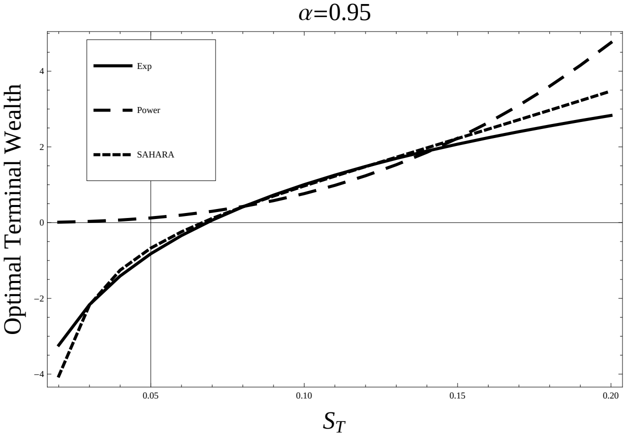
$$\beta \geq x_0 e^{(r-\frac{1}{2}(\lambda/\alpha)^2)T} / \sinh\left(-\alpha^{-1}(\lambda N^{-1}(\varepsilon)\sqrt{T}+\lambda^2 T)\right)$$

Critical α^* and β^*

α^*	β^*			
	$\varepsilon = 0.0005$	$\varepsilon = 0.001$	$\varepsilon = 0.005$	$\varepsilon = 0.05$
0.5	$0.23X_0$	$0.27X_0$	$0.39X_0$	$0.83X_0$
1	$0.85X_0$	$0.92X_0$	$1.19X_0$	$2.16X_0$
2	$1.98X_0$	$2.14X_0$	$2.66X_0$	$4.61X_0$
3	$3.06X_0$	$3.29X_0$	$4.07X_0$	$7.00X_0$
4	$4.12X_0$	$4.43X_0$	$5.36X_0$	$9.38X_0$
5	$5.18X_0$	$5.56X_0$	$6.85X_0$	$11.74X_0$

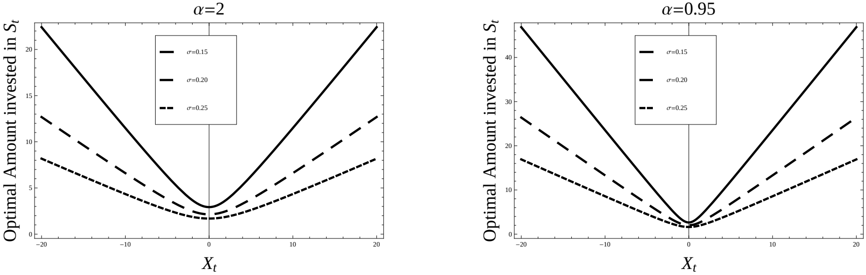
Critical α^* and β^* values with $r = 0.03$; $\mu = 0.08$; $\sigma = 0.15$; $T = 1$.

Optimal terminal wealth



Optimal terminal wealth for three utility functions as a function of S_T with $\alpha = 0.95$ and $\varepsilon = 0.01$.

Optimal investment amount in risky asset



Optimal investment amount in risky asset under SAHARA utility for $\alpha = 2$ (left) and $\alpha = 0.95$ (right) with parameter $\varepsilon = 0.005$.

Chen and Vellekoop (2014)

- Objective: finding explicit solutions for an economic agent who is allowed to end up with debts, i.e. a negative wealth at the end of the consumption and investment period.
  - Closed-form optimal policies of such agents have only been found for the specific class of exponential utilities
  - We formulate a model to study the consumption patterns for people who try to avoid debt, but do not avoid it at all costs.
  - A specific example which can motivate our approach is consumption as a result of sustained credit card debts.
    - Owning several credit cards simultaneously allows agents to consume beyond their means and may lead to situations in which their total wealth is negative

Chen and Vellekoop (2014)

- Standard investment & consumption model which assumes CRRA cannot be applied for such agents
- We assume
  - power utility for consumption:
  - SAHARA utility for terminal wealth:

⇒ Our problem describes a case between two extreme cases:

  - Power terminal wealth (people try to avoid debt at all costs)
  - Terminal wealth under cumulative prospect theory (risk seeking behavior)

- We assume a financial market with $d \geq 1$ traded risky assets and one risk free asset:

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j$$
$$\frac{dB_t}{B_t} = r_t dt,$$

  - $W_t := (W_t^1, W_t^2, \cdots, W_t^d)$ are d independent Brownian motions on our probability space $(\Omega, \mathbf{F}, \mathbb{P})$.
  - We fix a time horizon $T > 0$ and define for $t \in [0, T]$
  - The processes $r_t \geq 0$, $\mu_t = (\mu_t^1, \mu_t^2, \cdots, \mu_t^d)'$ and σ_t^{ij} which are adapted with respect to the filtration generated by the Brownian motions, which we denote by $\{\mathbf{F}_t\}_{t \in [0,T]}$

Optimal consumption and investment problem

- The optimization problem of the agent to choose an optimal portfolio and consumption rule to maximize

$$\sup_{(c,A):X^{A,c}\in\mathcal{X}(x_0)} \mathbf{E}\left[\int_0^{T\wedge\tau} \delta_s U_1(c_s)ds + \delta_T U_2(X_T)\mathbf{1}_{\{\tau>T\}}\right],$$

$$s.t.\mathbf{E}\left[\int_0^{T} H_t c_t dt + H_T X_T\right] = x_0$$

with $U_1(c) = K\frac{c^{1-\gamma}}{1-\gamma}$, (power utility) and $U_2(x) = U_{\alpha,\beta}(x)$, (SAHARA utility) for $K,\gamma,\alpha,\beta > 0$ and δ_s is a subjective discounting function.

$\Rightarrow$ Optimization problem is solved explicitly through dual approach

Mortality risk

- We assume that the time of death $\tau \geq 0$ of the agent can be modelled using a stochastic force of mortality process ς, which is adapted to a filtration $(\mathcal{G}_t)_{t\in[0,T]}$ that is independent of the Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and satisfies

$$\mathbb{P}(\tau \geq t|\mathcal{G}_t) := \zeta_t = \exp(-\int_0^t \varsigma_u du),$$

with $\varsigma_t \geq 0$ and $\int_0^t \varsigma_s ds < \infty$ for all $t \in [0, T]$.

Main results

Theorem

Assume that $\mathbb{E}[(\delta_T)^{-2/\alpha}] < \infty$. *Then there exists a unique constant* $c_0 > 0$ *such that the optimal consumption and terminal wealth are given by*

$$\begin{aligned} c_t^* &= c_0 \left(\frac{\delta_t \mathbb{E}[\zeta_t]}{H_t} \right)^{1/\gamma}, \\ X_T^* &= \beta \sinh\left(\tfrac{\gamma}{\alpha} \ln c_T^* - \tfrac{1}{\alpha} \ln K - \ln \beta\right). \end{aligned}$$

...Main results

Proposition

If the processes δ_t, $\Sigma_t^{-1}\theta_t$ and r_t and θ_t are deterministic then the optimal allocation process has the form

$$A_t^* = \left(\frac{1}{\gamma}f(t)c_t^* + \frac{1}{\alpha}\sqrt{(X_t^* - f(t)c_t^*)^2 + \beta^2 g(t)^2}\right)\Sigma_t^{-1}\theta_t$$

for certain positive deterministic functions f and g with $f(T) = 0$ and $g(T) = 1$.

Illustration: parameters

- We assume there are two tradable risky assets and one risk-free asset in the market. The asset price processes considered are

$$
\begin{aligned}
dS_t^1 &= S_t^1(\mu_1 dt + \sigma_1 dW_t^1) \\
dS_t^2 &= S_t^2(\mu_2 dt + \sigma_2(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2))
\end{aligned}
$$

- Parameters were chosen to be
$X_0 = 1$, $r = 0.05$, $\mu_1 = 0.08$, $\mu_2 = 0.10$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$,
$\rho = 0.3$, $T = 1$, $\alpha = 2$, $\beta = \frac{1}{2}X_0$, $K = 10$, $\gamma = 5$,
$\delta_t = \exp(-0.03t)$, $\zeta_t = \exp(-t/15)$.

Optimal consumption

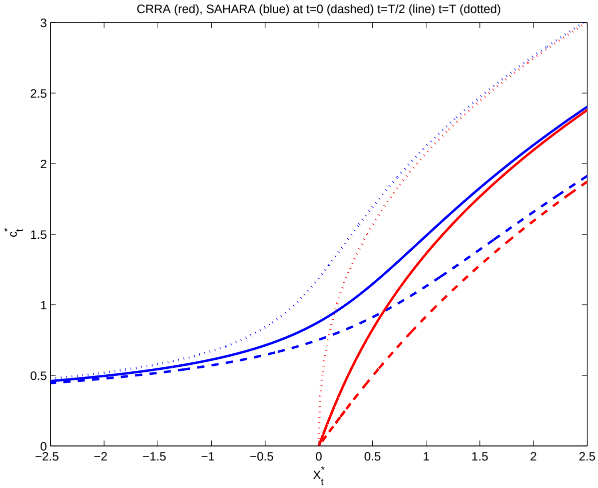


Figure : Consumption as function of wealth for CRRA and SAHARA terminal utility.

Optimal investment

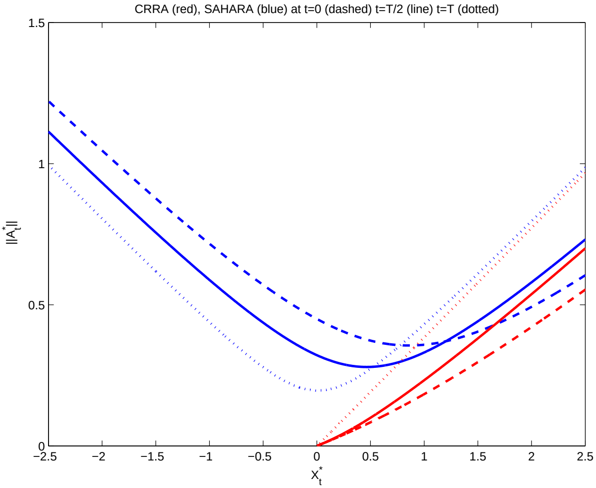


Figure : Investment as a function of wealth for CRRA and SAHARA terminal utility.

Optimal investment

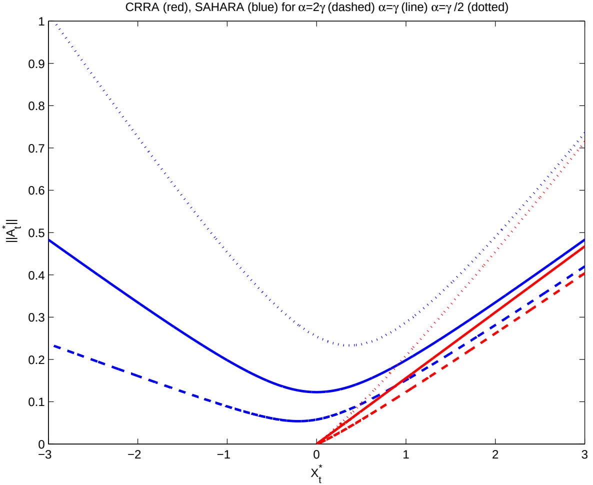


Figure : Investment as a function of wealth for CRRA and SAHARA terminal utility ($t = \frac{1}{2}T$).

Conclusion

- The present paper solves an optimal consumption and investment problem in a multi-asset financial market that explicitly incorporates negative terminal wealth.
  - CRRA preferences for consumption and SAHARA utility for terminal wealth.
  - Relying on the dual approach to stochastic optimization, we are able to give the solution for the optimal consumption, wealth and investment strategies.