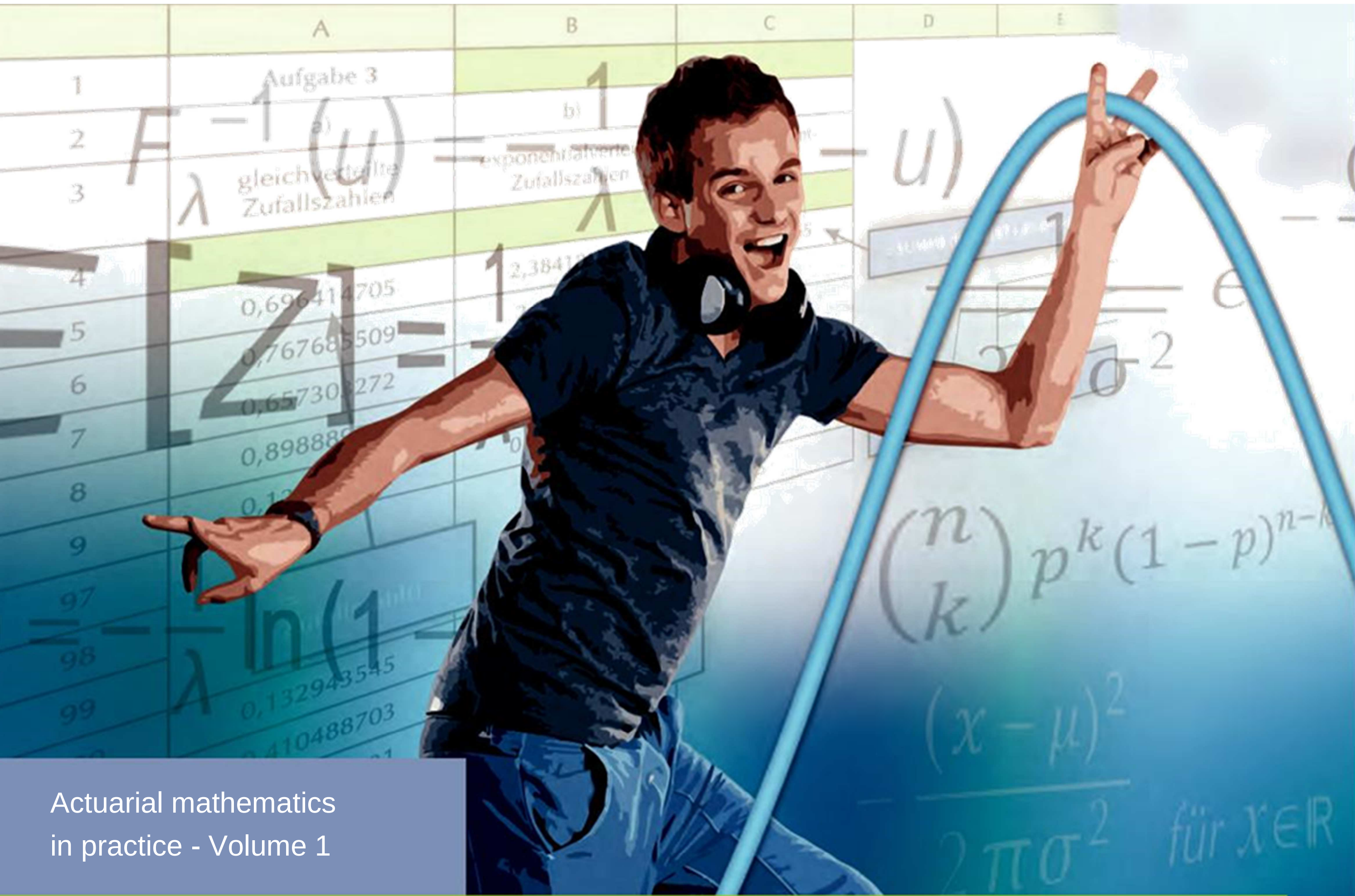

Actuarial mathematics
in practice - Volume 1

Stochastics. Simulation of property losses.

Teaching module with exercises and solutions
Sekundarstufe II

DAV
Deutsche Aktuarvereinigung e.V.

DGVFM
Deutsche Gesellschaft für Versicherungs- und Finanzmathematik e.V.

Actuarial mathematics in practice –
Volume 1

Stochastics.
Simulation of property losses.

Prof. Dr. Angelika May, Martin Oymanns
The authors would like to thank Nora Lisse and Thomas Adrian Schmidt for their cooperation.

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Contents

Introduction

Within this training brochure in actuarial mathematics, we are preparing an exciting medium for modern applied mathematics for school practice. The question of the insurability of risks has always been an important issue in society and occurs in a multitude of situations in our day-to-day lives. As a result, the majority of pupils have already come into contact with insurance companies, for example, when it comes to taking out liability insurance for their own scooter or health insurance for vocational training. Insurance is a self-evident part of our lives, often without any idea of the actual - especially mathematical - mechanisms behind the principle of insurance. This training booklet provides a basic idea of how the expected total losses of an insurance company, as well as an appropriate premium, can be calculated. In this context, property insurance is used as an example.

Edward Rowe Mores
(1731-1778) Coined the term first: actuary

Actuarial mathematics is a sub-area of mathematical stochastics. It is used daily by many actuaries in companies in the financial services sector. These are experts, who have mostly completed an additional, professionally oriented education following a mathematical degree in order to meet the special requirements in the respective areas of the insurance industry. For example, when designing an insurance premium, they ensure that the company can also bear the risks taken on from the customers. A variety of methods from probability theory and mathematical statistics are used.

The Stochastics Working Group has formulated seven key competences as the guiding principle "Data and Coincidence" for upper secondary education in its educational standards, three of which are explained in greater depth here:

The work group Stochastics of the Society for Didactics of mathematics involves representatives from universities, training seminars and schools for the improvement of the teaching of stochastics

- Modelling random events with probability distributions and their characteristic measures
- Use of simulations to gain approximate solutions for complex situations
- Basic phenomena concerning the law of large numbers.

The teaching of modern stochastics therefore aims to make decisions based on data and probabilities.

Results from mathematical didactic research prove that the frequently required complex application tasks "from practice" are, on the one hand, designed in a way in which pupils can use their knowledge about basic (stochastic) concepts and on the other hand, the tasks should conceptually complement the prior knowledge of the pupils. In this brochure we will address the issues of the relevance of tasks related to actuarial mathematics by providing the necessary background information on the subject of insurance, or by letting the pupils conduct their own research.

Prof. Dr. Angelika May
University of Oldenburg

Justification and relevance of the training booklet

The contents have been prepared for use by groups of students who have a basic knowledge of stochastics. Because of its compact and self-contained presentation, the training booklet is especially suitable for use in project weeks or for use by substitute teachers, as well as optional material for review at the end of a school year. Individual content, especially for stochastic simulation, can be chosen as a starting point for seminar papers. The topic is also suitable for mathematical working groups.

Structure

Based on the three core competences formulated in the introduction, this unit of instruction is structured as follows: So-called compensation in the collective and in time is essential for the proper functioning of insurance. This is guaranteed by the fact that many insured persons (belonging to the collective) pay premiums and losses do not occur at the same time. A basic prerequisite for the understanding of actuarial calculation is the relationship of the parameters (moments) of a random variable modelling the damage and the deterministic (i.e. non-accidental) premium to be paid by the policyholders. After some basic reflections in Module 1, we will discuss this in Module 2.

The expectation value and variance are also called first and second moments of a random variable.

In order to be able to calculate the expected value and the variance, assumptions must be made for the underlying distribution of the losses to be modelled, since this is usually unknown in advance. Unlike discrete distribution (e.g. in the case of a dice roll), these are continuous distributions. However, it is precisely the "popular" normal distribution that is not suitable. This will be discussed in Module 3. In addition, we will also present how Excel can be used to model the overall damage.

Random experiments, which have only a countable number of possible outputs, are called discrete.

Module 4 then shows how financially serious major losses can be mathematically taken into account.

The task sheets (with solutions) are intended to encourage the use of stochastic techniques, but also to establish the above-mentioned connection to the conceptual world of insurance.

In the appendix, the most important basic concepts of probability theory are explained, as well as some probability distributions and their properties.

1. The principle of insurance

1.1 Small effect, large loss

In general, (property) insurance covers future losses, which are currently unknown and which potentially endanger the solvency of the insurance company. An introductory example, from the area of private liability insurance, illustrates what is at stake:

Whilst on holiday, two children play on the roof of a school building with a burning tennis ball. Unfortunately, the ball ignites the roof of the school, which then burns down to the ground. Both survive the fire unharmed and, since it was a Sunday, there were no other injuries. But there was considerable material damage:

Rebuilding the school costs roughly € 6 million; Each of the two children would therefore have to pay € 3 million each. If their parents earn € 3,000 per month and use their entire salary for repayment, both families would need about 1,000 months, which is about 83 years, to repay the damage.

At this point, the private liability insurance of the parents comes into play, which carries the damage. The premium is about €100 per year.

An insurance company can only pay high losses if it generates sufficiently high premium income. It is therefore a necessary prerequisite that a large number of insured persons have taken out an appropriate insurance policy with the company.

In order to carry the loss for one of the families,
€ 3,000,000 : € 100 (per year) = 30,000 (families)
have to carry the loss together.

This happens in reality as well. The fact that insurance companies, among others, can offer private liability insurance at a price which is affordable for insured persons shows that the principle of compensation in the collective and in time works, but certainly also that such large losses usually occur rather rarely (in private liability insurance).

For compensation in time, the risk is to be payed by insurance premiums accumulated over a longer period of time, in which no claims were made.

Insurance as exchange transaction 1.2

In general, the following can be derived from the example:

In principle, insurance involves two parties: the insured person (policyholder) and the insurance company. The policyholder signs a contract with the insurance company in order to protect himself financially against a risk.

The two parties thus exchange a risk with potentially high financial impact against a comparatively low premium. This exchange transaction is possible because, at the same time, many people sign a contract with the insurance company.

Insurance coverage:
A cash amount (the premium of the insured) is exchanged against a future payment of a random amount (the insurer's payment)

The compensation payment by the insurance undertaking will be made if a contractually agreed event is the cause of the loss. In property insurance, this payment is usually determined by the extent of the damage incurred. Since the future is unknown, the future damage is a random variable.

For the policyholder, it is more worthwhile to take out insurance if the premium to be paid is as low as possible in relation to the amount of potential damage that may be incurred.

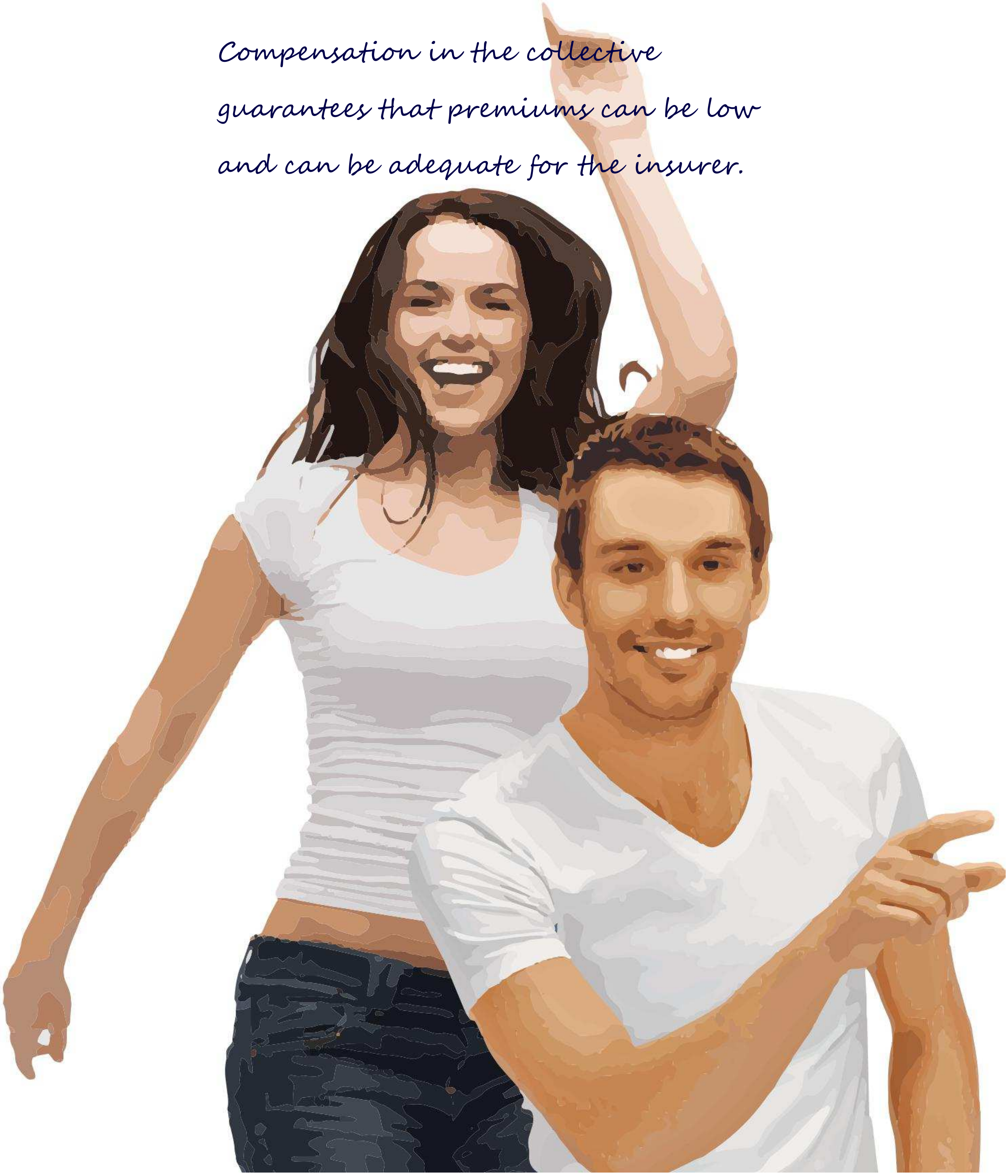
In this case, there is a certain inconsistency with the insurance company’s interests of charging a premium that is sufficient to cover all future payments. Premiums which are insufficient are, in the worst case, the ruin of the insurance company and would ultimately endanger the protection sought by the policyholder.

Furthermore, from today's point of view, the insurance company is unclear as to the future payments to policyholders. It is therefore important that the insurance company has a database that can be used to estimate future needs. We can therefore say that:

The future cash requirement is uncertain both in terms of the date of payment and of the level.
Nevertheless, the insurance company should be able to estimate it.

As already mentioned, the basic principle of risk compensation is used for this estimation. By combining several similar risks into one portfolio, the individual risks are balanced, so that the group is better able to bear individual losses. In fact, this idea of solidarity initially led to the precursors of modern day insurance, for example, as a burial union with the Phoenicians, with stonemasons in Egypt, or with the medieval guilds who provided their members with support in sickness and death.

Compensation in the collective guarantees that premiums can be low and can be adequate for the insurer.



Actuarial basics 2.

Compensation in the collective 2.1

How does the exchange of a fixed amount against a random variable work?

In the following, Z_i, for i = 1, ..., n, is the probability of the possible loss that can occur in a portfolio of n insured persons. In order to be able to determine a premium which covers the total annual losses as a result of all policyholders, we are firstly interested in the average value of the losses occurring in the portfolio.

Many insured persons who have insured themselves against similar risks with the same company form a portfolio.

$$\bar{Z} := \frac{1}{n} \sum_{i=1}^{n} Z_i$$

The law of large numbers helps us to determine this value for large n. In its empirical form it means that the mean value will converge to a constant value if the portfolio comprises sufficient numbers of people (the more people, the more accurate the approximation).

Don't forget:

The expectation value is linear, that is to say E[aX+bY] = a E[X] + b E[Y]

In this way, we are able to determine the overall losses that occur with great certainty in advance and determine the premium per policyholder on the basis of these considerations. From this we can derive the following result:

the first actuarial calculation principle:

The required premium B for a policyholder is equal to the expected loss, that is the expected value, $E[\bar{Z}]$, i.e.,.

$$B = E[\bar{Z}] = \frac{1}{n} \sum_{i=1}^{n} E[Z_i]$$

In addition to compensation in the collective, the compensation over time also provides a hedge against losses being reported.

In order to make the formulae as simple as possible, this aspect is not further regarded.

In other words, the policyholders themselves are liable for their expected loss. This phenomenon is referred to as compensation in the collective, that is, the law of large numbers assures the sufficiency of the principle of insurance.

Therefore, if we are able to determine the underlying distribution of the losses occurring in a portfolio, we can determine the annual loss to be expected by adding the expected losses per policyholder over the entire portfolio.

2.2 The expected value leads to certain ruin

Is B the premium which the insurer can actually charge from the insurer?

Unfortunately not, as another result from stochastics, **normal distribution approximation,** shows us:

It states that centralized (expectation equal to zero) and normalized (variance equal to one) sums of equally distributed random variables are approximately normal:

$$P\left(\frac{Z-B}{\sqrt{Var[Z]}} \geq \varepsilon\right) \approx 1-\Phi(\varepsilon) \approx 1-\Phi(0) = \frac{1}{2} \quad \text{for small} \quad \varepsilon > 0$$

For our case, this means that the required premium is suffice to cover incurred losses in only 50% of all cases! In other words, an insurance company that only collects the pure risk premium will certainly be "financially ruined" in the long term.

Safety loading 2.3

An adequate pricing therefore requires an extra charge on the pure risk premium, the safety loading. This results in

The second actuarial calculation principle:

The **risk premium** (P) for an insured is equal to the expected loss plus a **safety margin δ**, that is

$$P = E[\overline{Z}] + \delta$$

It may appear to be unacceptable to make a risk premium only dependent on the expectation value, and means that many medium-sized claims lead to the same premium as (in extreme cases) a large and a small claim. Nevertheless, the expected value as the reference value makes sense if there is no further information on the distribution of the risk.

*With the help of the central limit value theorem, a formula can be derived which directly establishes the relationship between the **probability of ruin** and the **safety margin**, which is often 30%*

If, however, the above-mentioned premium split for large and small claims should be included in the calculation of the premium, information on the spread of the risk is required.

The variance principle:

The **risk premium** (P) for an insured person is equal to the expected loss plus a **safety margin** a multiplied by the variance of the damage as a measure of the spread, that is

$$P = E[\overline{Z}] + a\, Var[\overline{Z}]$$

For the real premium calculation this means that the more accurate the loss information is, the more accurate the available data to be evaluated is. The expected value and variance can be determined particularly well if there is a plausible assumption about the distribution of the loss. Such distribution assumptions will play a central role in the teaching of modules 3 and 4.

*As a reminder, two random variables X and Y are called **stochastically independent** if for all events A and B:*
$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

In the variance principle example, we assume that the respective losses occur independently of each other and have the same distribution. This means that the associated random variables Z_i are independent and identically distributed. In particular, all Z_i have the same expectation value and the same variance.

Worksheet 1

Question 1:

Consider or investigate the risks to which your family is exposed and whether these risks are insurable.

Question 2:

Do you remember major insurance losses in recent times? Which? How high were the estimated losses or sums insured mentioned in the media? What can you find on the Internet if you search using the keyword "Wiehltal Bridge"?

Question 3:

Describe in your own words why insurance companies only work with compensation in the collective. What other conditions are required before we can refer to “insurance”?

Question 4: The cinema financing problem

Julia, Thomas and Antonio study maths and like to go to the cinema. They have occasional part-time jobs. However, this is not always possible so they often have no money for their weekly visit to the cinema. They therefore agree to insure themselves among themselves against the event "No money for the weekly visit to the cinema".

Discuss whether the following ideas define a meaningful insurance in the above sense or justify why a basic condition is violated:

a) All three pay an amount into a common cash pool, from which they replace any shortfall in paying for cinema tickets.

b) They persuade their fellow student, Arthur, who is always broke, and Martina, who always has enough money at her disposal, to join them. Do the five of them make up an insured collective?

c) In the last two years, they have often been unable to earn money during the holiday breaks. The three friends conclude that it is sufficient to insure themselves only for this time at the Xtra*Ltd. (London), which is always open to innovative ideas about money. In order to keep the premium low, they choose, among their 1,500 fellow students, those who receive just as much from their job as they do themselves. They are 139 students, of whom they can persuade 34 at Xtra*Ltd. to insure.

d) If you were Xtra*Ltd, would you offer Julia and her two friends this contract?

Stochastic Simulation 3.

Discrete and continuous probability distributions 3.1

In this module, we show how stochastic simulations of the annual loss S can be simulated in a portfolio of n insured persons. We will complete the chapter by following each step as an exercise in Excel.

Binomial distribution, one of the most important **discrete probability distributions**, describes the number of successes in a series of independent Bernoulli experiments. If the event "success" occurs with a probability of $p \in (0,1)$, then binomial distribution describes the probability that the event "success" occurs k times in n independent Bernoulli trials. This probability is given by

Jakob Bernoulli (1654-1705)

$$P(Z=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \textbf{for} \quad k=0,\ldots,n$$

where Z is equal to the number of successes. In order to produce a binomially distributed random variable Z, we first generate n randomly distributed random variables. We randomly draw numbers from the interval (0; 1), the draw of each number being equally probable. If the drawn number is less than p, the Bernoulli-distributed magnitude is 1, but if it is greater than p, we write the value 0. In this way we construct n randomly distributed random variables, that is, a sequence of N numbers that have the value 0 or 1. By summation, we then obtain our binomially distributed random variable Z.

In a Bernoulli experiment, there are only two possible outputs: "event occurs" and "event does not occur", often also with success "and" non-success "or simply with" 0 "and" 1 ".

To implement the method outlined above, we can use Excel, as we will see on the next sheet.

In order to generate random numbers from a **continuous distribution** with invertible (reversible) distribution function F, we use the **inversion method**.

In this case, first random numbers u are generated on the interval (0; 1), which are then inserted into the inverse function of the distribution function. The expected realisations x of the random variable X, which belongs to satisfy a continuous distribution with distribution function F, are calculated by

For reverse function F_λ^{-1} of strictly monotonic F_λ, is $F_\lambda^{-1}(F_\lambda(x)) = x$ for all x.

$$x = F^{-1}(u),\ u \in (0;1)$$

The random numbers generated in this way are required for the simulation of an insurance portfolio. For this purpose, we illustrate the inversion method again using **exponential distribution**.

This is a continuous probability distribution and it is concentrated on positive real numbers. A random variable Z follows an exponential distribution with parameters $\lambda \in \mathbb{R}_{>0}$, if it has the density function and the distribution function f_λ with

$$f_\lambda(z) = \begin{cases} \lambda e^{-\lambda z}, & z \geq 0 \\ 0, & z < 0 \end{cases} \quad \text{and} \quad F_\lambda(z) = \begin{cases} 1 - e^{-\lambda z}, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

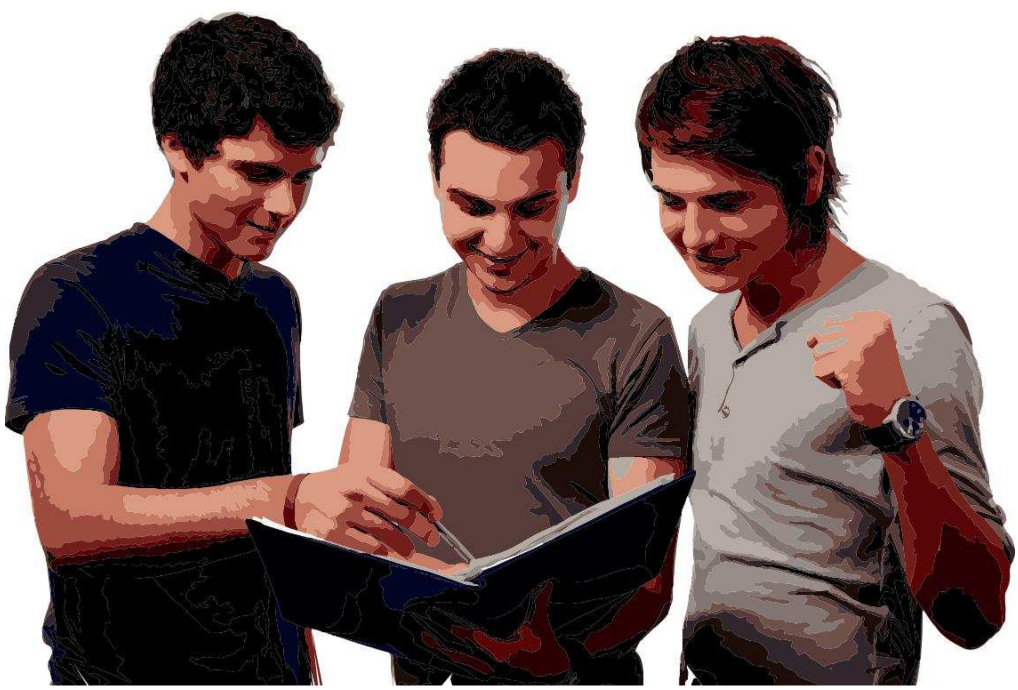
The expected value and variance in the exponential distribution are calculated as follows:

$$E[Z] = \frac{1}{\lambda} \quad \text{and} \quad Var[Z] = \frac{1}{\lambda^2}$$

For the inversion method, we need the inverse function F_λ^{-1} of F_λ. The exponential distribution results in

$$F_\lambda^{-1}(u) = -\frac{1}{\lambda} \ln(1-u) \quad \text{for} \quad u \in (0,1)$$

In the described way, of course, it is also possible to generate random numbers which satisfy other continuous distributions, always assuming the inverse function of the respective distribution function can be calculated.



The individual Model 3.2

In the following, we consider an insurance company that insures homogeneous risks. Furthermore, we assume that a total of n policyholders are insured with the insurance company, thus the size of the portfolio is n.

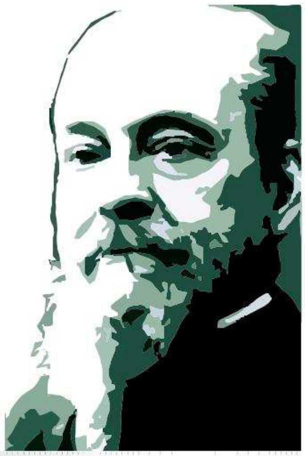
In the case of a homogeneous portfolio, the individual losses follow the same distribution and occur independently of one another.

The severity of the loss for the individual policyholders is again denoted by Z_i. In our model, only one loss per policyholder can occur per year. From the perspective of the insurance company, the random variable S, which describes the annual total loss of our portfolio of insured persons, can be represented as

$$S = \sum_{i=1}^{n} Z_i$$

At this point, two questions are particularly interesting for the insurance company:

- What is the distribution of the single losses Z_i ?
- What is the distribution of the aggregated loss per year S ?



Vilfredo Pareto (1848-1923)

The distribution of the individual loss severities can usually be estimated from the loss history using appropriate methods. We therefore assume in the following information that the distribution of the random variable Z_i is known.

When selecting the distribution, however, the severity of claims is always assumed to be positive since, of course, no negative loss can occur. For this reason, normal distribution is, for example, not suitable for modelling any losses. Exponential distribution, or so-called **Pareto distribution,** is often used in practice, especially in connection with large losses that can occur during natural catastrophes (see Module 4).

To determine the insurance premium using the variance principle, we only need the expected value and the variance of the annual total loss. As already mentioned, let us assume that the individual loss severities Z_i are independent and distributed identically like a loss Z.

The linear properties of the expectation value lead to the following equation:

$$E[S] = E\left[\sum_{i=1}^{n} Z_i\right] = \sum_{i=1}^{n} E[Z_i] = n \cdot E[Z_1] = n \cdot E[Z]$$

The expected value of the annual total loss results as the sum of the expected values of the individual losses, this means $n \cdot E[Z]$.

The variance of the yearly total loss can also be determined well if the variance of the individual loss is known. This is due to the independence of the individual loss severity

$$Var[S] = Var\left[\sum_{i=1}^{n} Z_i\right] = \sum_{i=1}^{n} Var[Z_i] = n \cdot Var[Z_1] = n \cdot Var[Z]$$

In the concrete case of exponentially distributed individual loss severity

(remember $E[Z] = \frac{1}{\lambda}$ *and* $Var[Z] = \frac{1}{\lambda^2}$ *)*

one derives the following for the total annual loss

$$E[S] = \frac{n}{\lambda} \quad \text{and} \quad Var[S] = \frac{n}{\lambda^2}$$

If the premium P for an insured person is calculated using the variance principle, one gets

$$P = \frac{1}{n}(E[S] + a \cdot Var[S]) = \frac{1}{\lambda} + \frac{a}{\lambda^2}$$

However, there is a crucial point we have not considered yet. Exponential distribution is not completely suitable for modelling losses because it assigns the probability 0 to the event "no loss occurs".

This of course does not correspond to the general observations in a real insurance portfolio. However, if we limit ourselves to the cases where a loss has actually occurred, we can still use exponential distribution for the amount of loss that has occurred.

However, this procedure means that we now no longer consider a fixed number n of insured persons, but instead have to take into account a random number N of losses that have occurred within one year within this portfolio. Therefore, in addition to the distribution of the severity, we must now also simulate the number of claims. This extension leads us to the **collective model** in the next chapter.

The collective model 3.3

We retain all the requirements from the previous model, except that we now introduce a further random variable *N*, which indicates the total number of losses occurring in one year in our portfolio of n insured persons. The probability that an insured person incurs a loss / makes a claim is $p \in (0;1)$ and is Bernoulli-distributed according to the considerations in the previous teaching module. We are now interested in the number *N* of claims actually occurring in the portfolio. This leads us again, as already shown, to binomial distribution. The probability magnitude of the individual damage is now denoted by the random variable Z_i.

This allows us to reformulate the collective losses for the collective model

$$S = \sum_{i=1}^{N} Z_i$$

where *N* is not a fixed number this time but a binomially distributed random variable, which is assumed to be independent of the individual severity Z_i.

In summary, we can note that the difference between the individual model and the collective model is that, in the former, the sum has a deterministic number of summands and zeros ("damage with height 0") can occur as summands, whereas in the collective model only the actually occurred losses and therefore the sum has a random number of summands.

Remember:
For binomial distribution with parameter n and p the expected value is equal to n p and the variance is equal np(1–p).
(see appendix)

Therefore, in the collective model two probability distributions influence the distribution of the annual total loss: the distribution of the number N of losses and that of the severity Z, which we have assumed to be exponentially distributed. For the expected value and variance of the annual total loss, this results in a somewhat more complicated representation:

$$\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[Z] \quad \text{and} \quad \mathrm{Var}[S] = \mathrm{E}[N]\,\mathrm{Var}[Z] + \mathrm{Var}[N]\,(\mathrm{E}[Z])^2$$

Both formulae are well known as **Wald Equations** (Abraham Wald 1902 – 1950).
In our specific case, this results from direct insertion:

$$E[S] = \frac{np}{\lambda} \quad \text{and} \quad Var[S] = \frac{np}{\lambda^2} + \frac{n(1-p)}{\lambda^2} = \frac{n(2-p)}{\lambda^2}$$

Here, too, we can determine the expected value and the variance of the annual total loss. This again is sufficient to calculate the premium using the variance principle:

$$P = \frac{1}{n}\left(E[S] + a \cdot Var[S]\right) = \frac{p}{\lambda} + \frac{ap(2-p)}{\lambda^2}$$

Worksheet 2

Question 1:

We begin with the simulation of a binomially distributed random number:

a) Create in Excel n = 100 uniformly distributed random numbers u_i on the interval (0;1).

Hint: The predefined function RAND() is helpful.

A random number x distributed to X is a realisation of the random variable X.

That is, the probability of obtaining a particular random number x is given by the distribution of X.

b) Define in a new column a new function that does the following:
Is $u_i \leq 0{,}1$ (generally: p), then a 1 is entered in the column;
is $u_i > 0{,}1$, then a zero 0. **Hint: Here the function IF() helps.**

The random numbers x_i obtained are Bernoulli-distributed.

c) Calculate $x = \sum_{i=1}^{n} x_i$ this way you have generated a Binominally-distributed random variable.

Question 2:

Now we generate exponentially distributed random numbers:

a) Create again as in task 1a) a on the interval (0,1) uniformly distributed random number u.

b) Put the resulting number u in the function $F^{-1}(u) = -\frac{1}{\lambda}\ln(1-u)$, in which $\lambda = 0{,}5$.

As a result, we obtain 100 exponentially distributed random numbers y_i, for $i = 1, ...,100$.

Question 3:

Finally, we calculate the annual loss S for a homogeneous portfolio of 100 insured persons as described in the teaching module for the collective model.

a) For this purpose, model a binomially distributed random number x, as in Task 2, and 100 exponentially distributed random numbers y_i , *for* $i = 1, \ldots,100$.

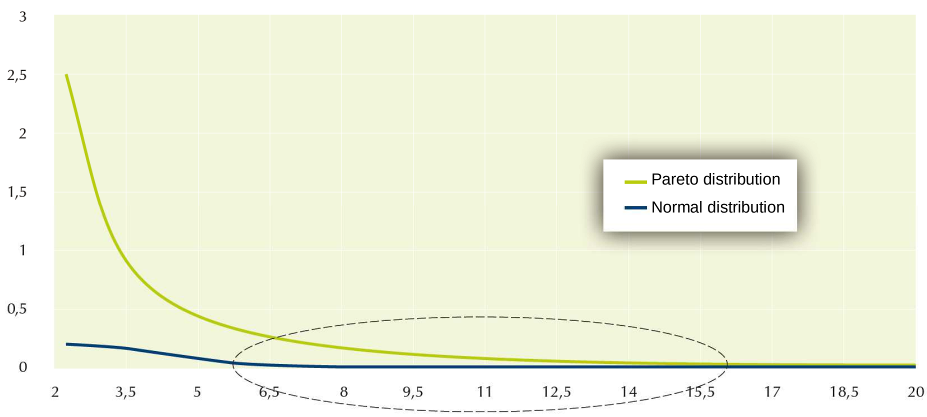
b) Calculate the annual total loss $S = \sum_{i=1}^{x} y_i$.

Stochastic simulation using the example of large losses 4.

Distribution assumption 4.1

In Training Module 1, we described the example of a large loss of € 6 million. In the subsequent teaching modules, we then learned about the importance of the correct assumed loss distribution for the feasibility of the premium. Large losses occur rarely, but can account for up to 90% of the loss. The precise modeling of such losses is therefore of particular importance.

For this purpose, distributions that are particularly well suited are those that assign for losses with a very high amount a positive probability, and which do not go too quickly towards zero. Such distributions have so-called **heavy tails**. This results in a further argument against the use of distributions whose density function resembles, for example, the Gaussian bell curve of normal distribution. These distributions decrease too quickly for large x and therefore systematically underestimate large losses.



On this basis we can now derive a common definition for distributions with heavy tails: For large x, the graph of the density function must be above the graph of the exponential function (cf. Teaching Module 3). Distributions for which this is not fulfilled are more suitable for modeling small losses. The exponential distribution helps us with the following definition:

Distribution of large losses:

We call a risk a "dangerous risk" if, for its density function, the following inequality is valid:

$$f(x) \geq e^{-ax} \quad \textit{for large } x \ (x \to \infty)$$

In practice, Pareto distribution is used for modeling **loss severities**, especially if large losses can occur. The density function f with the parameter α, and c >0, is defined as

$$f(x) = \begin{cases} \frac{\alpha}{c}\left(\frac{c}{x}\right)^{\alpha+1} & \text{for } x>c \\ 0 & \text{for } x\leq c \end{cases}$$

The expectation value is finite if $\alpha > 1$; the variance is finite for $\alpha > 2$. Pareto distribution is therefore particularly suitable for risks with a large variance and is also used in practice, for example, in fire insurance.

The **number of losses**, on the other hand, has previously been taught as a binomially distributed random variable. For sufficiently large n, according to Poisson's theorem, binomial distribution converges to **Poisson distribution**, where $n \cdot p \approx \lambda > 0$. The density function of Poisson distribution is

$$p_k = P(N=k) = e^{-\lambda}\frac{\lambda^k}{k} \quad \text{for} \quad k \in \mathbb{N}_0$$

We assume in the following that the number of losses is Poisson-distributed with parameter λ.

Simulation of number of losses and their severity 4.2

In Teaching Module 3, we saw how we can simulate the two random variables of the collective model if they are distributed binomially or exponentially. We will now learn about Poisson and Pareto distribution.

Simeon Denis Poisson (1781-1840)

Simulation of Poisson-distributed number of losses with parameter λ

We apply the following algorithm:

1. We start with $n := 0$ and $T := 1$.
2. Then we simulate a (0;1) uniformly distributed random variable with realisation u and set $T := uT$.
3. If $T \geq e^{-\lambda}$, we set $n := n + 1$ and go back to step 2 .
4. If $T < e^{-\lambda}$, n is a realisation of the number of losses N.

Simulation of Pareto-distributed loss severity

If Z has a Pareto distribution with parameters α and c, then Z is distributed as $c\,U^{-1/\alpha}$, where U is a random variable which is uniformly distributed to (0; 1).

The total loss of a compound **Poisson- Pareto distribution** is now calculated using the following algorithm:

1. Initially we generate a realisation n of the Poisson-distributed number of losses N *as above*.
2. Then we simulate n independent realisations $z_1, \ldots, z_n$ of the loss severity Z. This is, according to previous considerations, distributed identically to all of the z_i.
3. Finally the sum of the realisations of $z_1, \ldots, z_n$ from 2. provides a realisation of the total loss S.

The difference to the simulation from teaching module 3 is, firstly, that we consider a distribution with heavy tail (and thus leads to a heavy-tailed distribution of the total loss), and secondly, assuming a compound Poisson-Pareto distribution of the losses.

For training purposes, this procedure is justifiable for the modeling of a total loss, in which major losses are also taken into account. Since this method is very time-intensive in the case of large insurance portfolios, in reality compound Poisson-Pareto distribution is approximated with normal distribution. This makes simulation significantly less time-intensive - but may permit approximation errors.

Worksheet 3

You are employed as an actuary for a large insurance company, which offers various insurance products mainly for industrial companies. Your task is developing new tariffs in property insurance.

Market analysis has recently shown that there is a promising market for fire insurance for large companies. This is advantageous since you already have some experience with regard to the number of losses N and the individual severities Z_i in this field. So you know that although very high losses can occur, these are rather rare. The loss Z is therefore in a rather low range.

Therefore, you are now asked to develop appropriate insurance pricing. From experience, you know that there is a need for such insurance for around 100 large companies.

Question 1:

Determine a possible annual total loss S according to the algorithm given in teaching module 4. Use Poisson (λ = 50) and Pareto distribution (c = 100, α = 10).

What is the simulated loss?

Hint: This task is largely to be solved as in the second worksheet; For the distribution of Pareto, the relation to the uniform distribution given in Module 4 is to be used.

Question 2:

Calculate the expected value of the annual total loss P. How close is it to your calculated loss?

Hint: Use the formula for the expected value of the annual total loss in module 3; The expected value of a Poisson-distributed random variable is *E[N]* = λ; For a Pareto-distributed random variable:

$$E[Z] = \left(\frac{\alpha}{\alpha - 1}\right) \cdot c$$

Question 3:

Finally, you want to determine the annual premium payable by each large company. To do this, change the expected loss to all clients and add a 30% safety margin. Is the total premium income sufficient to cover the actual loss simulated in task 1?

Solutions for worksheet 1

Question 1:

Possible insurable risks are for example

Illness **(Health insurance)**	Car accident **(Own damage, third party liability insuance)**	Fire in flat **(Residential building, household ins.)**
Theft of a racing bike **(household insurance)**	Accident on your way to school **(Accident insurance)**	Disability, etc.

Question 2:

Recent catastrophes

Occurence	Date	Estimated loss
Tsunami and Earthquake, Japan	March 2011	€ 210,0 billion
Flood, Australia	December 2010 / January 2011	€ 7.3 billion
Hurricane Katrina, North Carolina, USA	August 2005	€ 125,0 billion

Source: Munich Re; GeoRisikoForschung

On 26 August 2004, a tanker fell from the motorway bridge "Wiehltal" as a result of an accident and burned completely underneath the bridge. As a result of the fire the bridge was so badly damaged that a loss of € 30 million arose. This accident is one of the biggest car liability claims in German history.

Question 3:

Due to the fact that the expectation is used as a reference for the premium, some policyholders pay more, others less than they need themselves. If the collective size is too small, either the premium is unacceptably high, or the insurance company can no longer manage the loss caused very quickly due to lack of capital. Other prerequisites are: compensation over time (over several years), similar risk and similar threat.

Question 4:

a) In principle a good idea (common danger: no money available, common interest in the cinema), but the collective is (too) small. This could quickly exhaust the cash pool.

b) No. Arthur is already bankrupt; He has no money to lose. Martina is also not at risk because she always has enough money at her disposal.

c) All have the same amount of money from their side jobs, so they form a very homogenous collective as well as a larger (and therefore easy to estimate) collective. Since the time for which the students want to insure themselves is very risky, the necessary compensation in time does not happen. Therefore, the risk for Xtra*Ltd. to run out of money is high. That is why the company would rather give up the business.

Solutions for worksheet 2

Since this work sheet is essentially designed with random numbers (which will be different in each case), we will restrict ourselves to specifying the working out:

Question 1:

Note: The solution to these Questions can, of course, also be worked out in a single table. The sample solution is also available on the internet https://aktuar.de/aktuar-werden/fuer-die-schule/Seiten/default.aspx

	A	B	C
1	Task 1		
2	a)	b)	c)
3	Uniform distributed numbers	Bernoulli distributed numbers	Binominal distributed numbers
4			
5	0,175732415	0	10
6	0,319989058	0	
7	0,978466233	0	= sum (B5:B104)
8	0,79873209	0	
9	0,092363317	1	
…			
…			
…			
100	= RAND()	= IF (A5<0.1, 0, 1.0)	
101			
102	0,227719299	0	
103	0,181501373	0	
104	0,951120051	0	

For the solution we proceed as shown in the picture on the left. In column A, we use the function RAND() to generate 100 uniformly distributed random numbers. In the next column we use the function IF() to compute the case distinction required in (1b), which then supplies 100 Bernoulli-distributed random numbers. The random sum in cell C5 is finally binomially distributed.

Question 2:

From each of the uniformly distributed random numbers from task 1a) (e.g. 0.1757;. see cell A5), we use the formula

$$F^{-1}(0{,}1757) = -\frac{1}{0{,}5}\ln\left(1-0{,}1757\right) = 0{,}3864$$

to calculate an exponentially distributed random variable.

Question 3:

Again, in the first column we produce 100 uniformly distributed random numbers. If we proceed in a similar fashion to task 2, we get in column B exponentially distributed random numbers.

	A	B	C	D	E
1	Task 3				
2	a)	b)	c)		
3	Uniform distributed numbers	Exponential distributed numbers	Annual total loss		
4					
5	0,696414705	2,384185339	23,4856755		
6	0,767685509	2,91932652			
7	0,657303272	2,141818794			
8	0,898889758	4,583087713			n
9	0,121015145	0,257975223			10
97			λ		
98			0,5		
99					
100	0,132943545	0,285302377			
101	0,410488703	1,0			
102	0,201093131	0,			
103	0,286208478	0,67432869			
104	0,987557005	8,773194921			

= SUM (INDIRECT („B5:B“ &(E9+4)))

= binomially distributed sum from 1c)

= RAND()

= -1/$C$98* LN(1-A5)

Then, in cell C5, we can finally compute the desired sum *S* by adding one of the exponentially distributed random numbers from column B in column C randomly. The number of the random numbers to be added is arrived at using the binomially distributed random variable from task 1c)

Solutions for worksheet 3

Question 1:

To simulate a year's total loss, we use a similar method to that used to solve Worksheet 2:

	A	B	C	D	E	F	G
1	Task 1						
2	In this column uniformly distributed random numbers are generated for Z	In order to guarantee independence of Z and T, uniformly distributed random numbers are generated for T.	Generation of Pareto distributed random numbers by regarding the hints in chapter 4 (Realisation of Z).	Step 1 & 2 of the algorithm in chapter 4: Generation of the simulated values of the Poisson distributed random	Step 3 and 4 of the algorithm: Checking, whether T>=exp(-λ)	Generation of realisations of the Poisson distributed random variable n (Step 4 of the algorithm)	Calculation of the aggregated yearly loss as random number with n summands (corres-ponding to the values in cell F4 and the following), Summands from column C
3							
4	0,382948292	0,487206779	110,0743137	0,487206779	0	53	5915,650882
5	0,867430652	0,300596368	101,4	0,146452588	0		
6	0,84990556	0,590149774	101,6	0,086428962	0		
7	0,828194927	0,742772811	101,9029469	[illegible]	0		
8	0,807093043	0,904066014	102,1662939	[illegible]	0		
9	0,861290209	0,14038249	101,5044422	0,008147572	0		
…			…	…	…		
…			…	…	…		
…	…	…	…	…	…		
99	0,512460112	0,790033805	106,913856	6,49362E-42	1		
100	0,530309728	0,994717789	106,5484265	6,45932E-42	1		
101	0,321127361	0,181517314	112,0294668	1,17248E-42	1		
102	0,39758171	0,546067407	109,6623025	6,40253E-43	1		
103	0,759217612	0,859137173	102,7929601	5,50065E-43	1		
104							
105							
106							
107							

= B4

= B5 * D4

=($F$106)*A4^(-1/$F$107)

= 100 –
Sum (E4:E103)

= IF(D4 >
EXP(-$F$105);0;1)

= sum (INDIRECT („C4:C" &(F4+3)))

(The function „INDIRECT" let's sum up the first n entries in column C – for arbitrary n)

= RAND()

λ = 50
c = 100
α = 10

First, we generate 100 random numbers in columns A and B respectively. This is necessary to ensure the independence of the Pareto-distributed random numbers in column C and of the Poisson-distributed random number in cell F4. Thereafter the Pareto-distributed random numbers in column C are generated from the values in column A using the formula $cU^{-1/\alpha}$.

Together with the interim calculations in columns D and E, we can also obtain from the values in column B a Poisson-distributed random number (cell F4). Hence, the total annual loss in cell G4 can be calculated. In the present case – the result varies depending on the generated random numbers in column A and B - a simulated annual total loss of around € 5915.65 arises.

Solutions Worksheet

Question 2:

According to Chapter 3, the expectation of the annual total loss applies:

From $E[N] = \lambda = 50$ and $E[Z] = \left(\frac{\alpha}{\alpha - 1}\right) \cdot c = \left(\frac{10}{9}\right) \cdot 100 = 111{,}11$ follows

$E[S] = E[N] \cdot E[Z] = 5555{,}55.$

The insurance company must expect a loss of approximately €5,556.

Note: The model answer is also available on the Internet https://aktuar.de/aktuar-werden/fuer-die-schule/Seiten/default.aspx

Question 3:

Taking into account the expected value calculated in Exercise 2 and the 30% safety margin, the premium P for each company is calculated

$$P = (1+0{,}3) \cdot \left(\frac{1}{100} E[S]\right) = 1{,}3 \cdot 55{,}\bar{5} = 72{,}22\,€$$

The total sum of the premium income of around € 7,222 is sufficient in our case to cover the loss damage simulated in Exercise 1. However, if the simulation is carried out several times, it is found that this is not always the case. Therefore, additional reserves are created in insurance companies for these cases.

Mathematical appendix

In the following, we will list the most important principles of probability theory that are relevant for the contents of this booklet.

A **Random variable** X is a function, which maps the sample space Ω onto the real numbers $\mathbb{R}$, this means $X: \Omega \to \mathbb{R}$, $\omega \mapsto X(\omega)$

If the value set $X(\Omega) = \{x \in \mathbb{R} : x = X(\omega)$ for a $\omega \in \Omega\}$ is countable, we say that X is a **discrete** random variable, otherwise (for the case $X(\Omega)$ if it is uncountable, e.g.,. $X(\Omega) = \mathbb{R}$) we refer to a **continuous** random variable.

For a discrete random variable X and for a real number x, the probability of the event $\{\omega : X(\omega) = x\}$ can be allocated. The corresponding functional rule

$$W: x \mapsto P(X = x) := P\{\omega : X(\omega) = x\}$$

is called **density function** of the random variable X. It is also said that the random variable X is distributed to W.

Distributions are characterized by the (cumulative) **distribution function** F or (for continuous random variables) by their **density function** f. The following connection is important

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

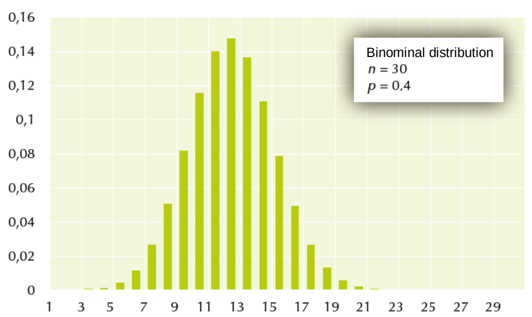
A random variable has different characteristic indicators, which are determined by the random variable's distribution. Particularly prominent are the first two moments, expected value E (X) and variance Var (X). In normal distribution, for example, the two parameters directly yield the expected value (equal) as well as the variance (equal to Ω^2). The expected value and variance for a random variable X with event space are calculated according to the following formulae, depending on whether the underlying random variable is continuous or discrete.

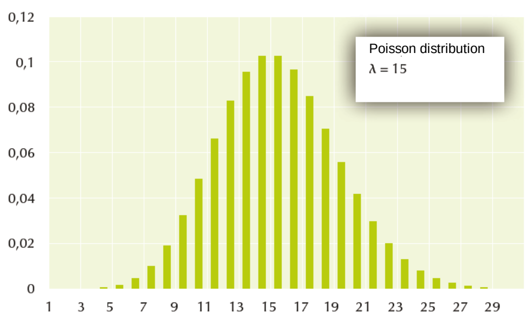
Moments	Expected value	Variance
discrete	$E[X] = \sum_{x \in X(\Omega)} x P(X = x)$	$Var[X] = \sum_{x \in X(\Omega)} (x - E[X])^2 P(X = x)$
continuous	$E[X] = \int_{X(\Omega)} x f(x) dx$	$Var[X] = \int_{X(\Omega)} (x - E[X])^2 f(x) dx$

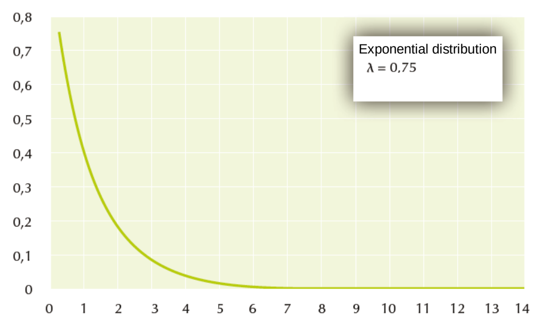
At an intuitive level in the context of random experiments, the expected value can be thought of as the mean value, i.e., the average value of the random variable per experiment in the long run. The variance (or the standard deviation as the square root of the variance) then indicates how much the values of the random variable spread around the expected value.

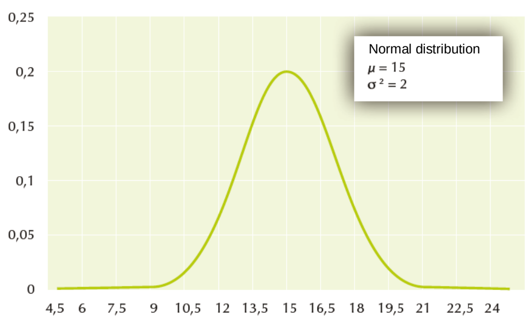
Various distributions are relevant in an insurance context, for example:

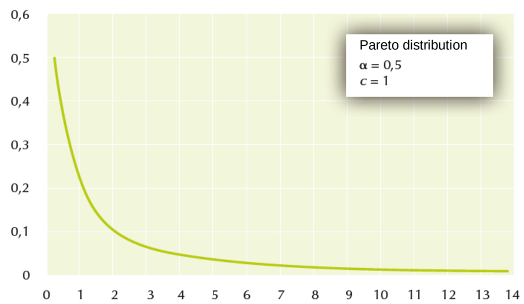
Distribution	Density function	Expected value	Variance
Binominal distribution (discrete)	$\binom{n}{x} p^x (1-p)^{n-x}$ *für* $X \in \{0,1,2,\dots,n\}$	np	$np(1-p)$
Poisson distribution (discrete)	$e^{-\lambda}\frac{\lambda^x}{x!}$ *für* $x \in \mathbb{N}_0$	λ	λ
Normal distribution (continuous)	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ *für* $x \in \mathbb{R}$	μ	σ^2
Exponential distribution (continuous)	$\lambda e^{-\lambda x}$ *für* $x \geq 0$ 0 *sonst*	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Paretol distribution (continuous)	$\frac{\alpha}{c}\left(\frac{c}{x}\right)^{\alpha+1}$ *für* $x > c > 0$ 0 *sonst*	$\frac{\alpha c}{\alpha-1}$	$\frac{\alpha c^2}{(\alpha-2)(\alpha-1)^2}$











Mathematics in insurance companies

Actuaries: Mathematical experts in the insurance business

In this training brochure, we have used an example of practice that is not very far from the mathematical calculations that must be made to develop insurance tariffs. The required calculations are carried out in insurance companies by mathematicians. In reality, however, a number of other aspects have to be considered. Therefore, prior to the actual development of an insurance tariff, market analysis is usually carried out using statistical methods in which the needs of the potential customers are estimated as precisely as possible in line with a certain insurance protection. For the concrete calculation of the tariff, so-called "appropriate actuarial assumptions" have to be used to determine how high the monthly or annual premiums have to be in order to ensure sufficient reserves for the indemnity of all policyholders in the case of a claim.

The actuarial equivalence principle applies: "Expected premiums are expected to be equal to expected payments".

Once a tariff has been calculated and distributed to customers, the work of the actuaries is not yet over, however.

For example, the financial reserves determined at the start of the contract have to be continuously checked to ascertain whether they are still sufficiently high enough to service the insurance contract. This ongoing check / monitoring is very important as, for example, endowment (life) insurance policies may, in some cases, run over many decades.

During the term of the agreement, an insurance company receives payments from its customers on an ongoing basis, which are often paid back at a much later date. Monitoring current and future investments by term and by achievable return is therefore also an important task.

The term "primary insurance" covers all insurance companies which provide persons or companies with insurance cover.

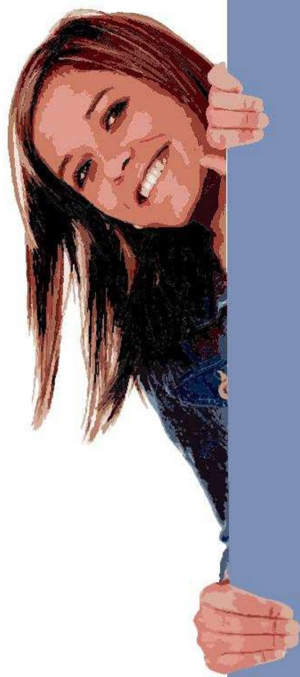
Mathematicians, who work on the above-mentioned tasks for insurance companies, are called actuaries. They are experts, who have been specially trained in the various actuarial disciplines of probability theory and statistics. With their expertise, they ensure that the insurance contracts signed can always be fulfilled even in the case of very long maturities of 30, 40 or 50 years, which are usual in the area of old-age provision. Actuaries therefore stand for high customer protection.

Job description

Actuaries work in all sectors of the insurance industry. They can be found in insurance companies, pension funds, advisory firms, the German/international supervisory / regulatory authorities for financial services and n relevant associations.

Reinsurers insure exclusively insurance companies, covering a part of the risks assumed by the primary insurers.

As already mentioned, actuaries are delegated with a variety of tasks. In addition to classical pricing, they calculate the financial reserves required by the company, advise corporate management on meaningful concepts for the various types of insurance and evaluate them. In addition, they document these processes, for example, for the supervisory authorities. They develop strategies for the secure investment of the insurance companies' financial resources or, in risk management, monitor and manage the various risks arising from the wide range of insurance contracts concluded, as well as the companies' individual investments. Often, they work interdisciplinary, which means regular communication with lawyers, operations, auditors and IT is an important part of their work.



Education

For a future life as a professional actuary, the study of mathematics or mathematical economics provides an excellent preliminary training basis. Many universities have also set up bachelor's and master's degree programs, which focus on training in insurance and financial mathematics. Besides the most important knowledge of probability theory and statistics, the basics of business management are included as well. In order to obtain the necessary knowledge for the everyday work environment, you have the possibility to complete an apprenticeship as an actuary. The training usually takes three to four years and it provides a comprehensive knowledge of insurance and financial mathematics. In addition, there is a specialisation in a particular subject, so that all the more advanced methods of your own chosen sphere of activities can be studied.

As is the case with most professions, continuous professional development training is of course also essential after completion of the apprenticeship and there are numerous development possibilities, ranging from self-study to expert conferences (where over one thousand actuaries from all over Germany meet twice a year).

Profession of the future

Due to their special expertise, actuaries are now utilised not only in the development of insurance products, but also in many other positions within the financial services industry. Therefore, they are sought-after specialists and the profession offers a lot of variety, good career prospects and very good salary prospects. The profession has traditionally been regarded as a man's domain but now however, more women than men are undertaking actuarial training in Germany

Actuary: Versatile job with very good career prospects and an attractive salary.

As insurance companies are increasingly offering their services all over Europe, there is also the possibility of an international career. Through new, joint financial supervision regulations, further exciting areas of responsibility have been created that offer new challenges and opportunities for actuaries as well.

Actuaries are members of a professional body, the German Association of Actuaries (DAV), which had more than 4,400 members at the beginning of 2015. Additionally, around 2,200, mainly junior financial and insurance mathematicians are currently taking part in regular actuarial training programmes.

We create new knowledge

DAV
DEUTSCHE AKTUARVEREINIGUNG e.V.

DGVFM
DEUTSCHE GESELLSCHAFT FÜR VERSICHERUNGS- UND FINANZMATHEMATIK e.V.